122. Floquet Theory

Applicable to Linear ordinary differential equations with periodic coefficients and periodic boundary conditions.

Yields Knowledge of whether all solutions are stable.

Idea If a linear differential equation has periodic coefficients and periodic boundary conditions, then the solutions will generally be a periodic function times an exponentially increasing or an exponentially decreasing function. Floquet theory will determine if the solution is exponentially increasing (and so “unstable”) or exponentially decreasing (and so “stable”).

Procedure Suppose we have an nth order linear ordinary differential equation whose coefficients are periodic with common period $T$. The general technique is to write the ordinary differential equation as a first order vector system of dimension $n$ (see page 146), and then solve this vector ordinary differential equation for any set of $n$ linearly independent conditions, for $0 \leq t \leq T$.

This yields a propagator matrix $B$, such that $y(t + mT) = B^m y(t)$, where $m = 1, 2, \ldots$. Hence, to determine the stability of the original problem, we need only determine the eigenvalues of $B$. If any of the eigenvalues are larger than one in magnitude, then the solution is “unstable.”

As an example of the general theory, we consider second order linear ordinary differential equations of the form

$$y'' + q(t)y = 0,$$  \hspace{1cm} (122.1)

where $q(t)$ is periodic with period $T$, i.e., $q(t + T) = q(t)$. We can write equation (122.1) as a vector ordinary differential equation in the form

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} \mathbf{y},$$

where $\mathbf{y}(0) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$ is known in principle. We now define $u(t)$ and $v(t)$ to be the solutions of

$$\begin{bmatrix} u(t) \\ u'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}, \quad \begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$  \hspace{1cm} (122.2)

and

$$\begin{bmatrix} v(t) \\ v'(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ v'(t) \end{bmatrix}, \quad \begin{bmatrix} v(0) \\ v'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  \hspace{1cm} (122.3)
Then, by superposition, \( y(t) = A(t)y(0) = \begin{bmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{bmatrix} y(0) \). Equivalently, \( y(T) = By(0) \), where \( B = A(T) \). Hence, \( y(2T) = By(T) = B^2y(0) \), \( y(3T) = B^3y(0) \), etc. The eigenvalues of \( B \) are needed to determine stability. By the usual calculation, \( \lambda \) will be an eigenvalue of \( B \) if and only if \( |B - \lambda I| = 0 \). We calculate,

\[
|B - \lambda I| = \begin{vmatrix} u(T) - \lambda & v(T) \\ u'(T) & v'(T) - \lambda \end{vmatrix} = \lambda^2 - \lambda[u(T) + v'(T)] + [u(T)v'(T) - u'(T)v(T)]
\]

(122.4)

where we have defined \( \Delta = u(T) + v'(T) \), and we set \( u(T)v'(T) - u'(T)v(T) \) equal to one because the Wronskian of equation (122.1) is identically equal to one. Solving equation (122.4) for \( \lambda \), we determine that

\[
\lambda = \frac{1}{2} \Delta \pm \sqrt{\frac{1}{4} \Delta^2 - 1},
\]

and so we conclude

- If \( |\Delta| < 2 \), then, for both values of \( \lambda \), we have \( |\lambda| \leq 1 \) and so all of the solutions to equation (122.1) are stable.
- If \( |\Delta| > 2 \), then there is least one value of \( \lambda \) with \( |\lambda| > 1 \) and so the solutions to equation (122.1) are unstable.

**Example**

Suppose we have the equation

\[
y'' + f(t)y = 0,
\]

(122.5)

where \( f(t) \) is a square wave function of period \( T \)

\[
f(t + T) = f(t) = \begin{cases} -1 & \text{for } 0 \leq t < T/2, \\ 1 & \text{for } T/2 \leq t \leq T. \end{cases}
\]

(122.6)

Note that \( f(t) \) is *not* continuous. This does not change any of the analysis. We can solve equation (122.5) and equation (122.6) by using \( f(t) = -1 \) and solving for \( \{u(t), v(t)\} \) in the interval \( 0 \leq t < T/2 \). Then we set \( f(t) = 1 \) and solve for \( \{u(t), v(t)\} \) in the interval \( T/2 < t \leq T \), using as initial conditions the values calculated when we took \( f(t) = -1 \). See the section on solving equations with discontinuities (page 264).

The solutions of equations (122.2) and (122.3) are found to be (for \( T/2 < t \leq T \))

\[
u(t) = (\sinh \tau \sin \tau + \cosh \tau \cos \tau) \sin t + (\sinh \tau \cos \tau + \cosh \tau \sin \tau) \cos t,
\]

and

\[
v(t) = (\cosh \tau \sin \tau + \sinh \tau \cos \tau) \sin t + (\cosh \tau \cos \tau - \sinh \tau \sin \tau) \cos t,
\]
where \( \tau = T/2 \). From these equations, we determine \( \Delta \) to be

\[
\Delta = u(T) + v'(T) = 2 \cosh \tau \cos \tau.
\] (122.7)

The conclusion is that the solutions to equation (122.5) will be stable or unstable depending on whether the magnitude of \( \Delta \), as given by equation (122.7), is greater than or smaller than 2. Different values of \( T \) will give different conclusions. For example,

- If \( T=17 \) or \( T = \pi^2 \), then \( |\Delta| > 2 \) and some unstable solutions to equation (122.5) exist.
- If \( T=1 \) or \( T = \pi \), then \( |\Delta| < 2 \) and all to the solutions to equation (122.5) are stable.

Notes
1. Mathematicians call this technique Floquet theory, whereas physicists call it Bloch wave theory. Solid state physicists use this technique to determine band gap energies.
2. Note that the periodicity of \( f(t) \) in equation (122.5) does not, by itself, insure that \( y(t) \) has a periodic solution. If, however, \( f(t) \) is periodic and has mean zero, then equation (122.5) will have a periodic solution of the same period.
3. The linear system \( y' = B(t)y \) is said to be noncritical with respect to \( T \) if it has no periodic solution of period \( T \) except the trivial solution \( y = 0 \). Otherwise, the system is said to be critical.
4. See also Coddington and Levinson [1, pages 78–81], Kaplan [3, pages 472–490], Lukes [5, Chapter 8, pages 162–179], and Magnus and Winkler [6, pages 3–10].

References